I: Dirichlet's Principle

The mathematical result is based on the physical idea of energy. It states that all C^1 functions $w(\mathbf{x})$ in Ω that satisfies Dirichlet's boundary conditions

$$w = h(\mathbf{x}) \quad \text{on } \partial\Omega \tag{1}$$

the lowest energy occurs for the harmonic function in Ω satisfying

$$\nabla^2 w = 0 \qquad \text{in } \Omega \ . \tag{2}$$

The energy is defined as pure potential energy, that is, $E[w] := \frac{1}{2}\int_\Omega |\nabla w|^2 d\mathbf{x}$, since there is no motion associated with the principle, hence no kinetic energy. This corresponds to the general principle in physics that a system prefers going into a state of lowest energy (the 'ground state'). More precisely, let $u(\mathbf{x})$ be the unique (harmonic) function in Ω that satisfies (1), (2). Let $w(\mathbf{x})$ be any differentiable function defined on Ω satisfying the boundary condition (1) and for which $E[w]$ makes sense. Then the principle gives $E[w] \geq E[u]$.

Let us go through an argument for why this happens. Let u be a function that satisfies (1), (2) and minimizes E. Let v be any differentiable function defined on Ω that vanishes on the boundary of Ω. Then $u + \varepsilon v$ satisfies (1). So, if the energy is smallest for u, we have

$$\begin{aligned} E[u] \leq E[u+\varepsilon v] &= E[u] + \varepsilon \int_\Omega \nabla u \cdot \nabla v d\mathbf{x} + \varepsilon^2 E[v] \\ &= E[u] - \varepsilon \int_\Omega v \nabla^2 u d\mathbf{x} + \varepsilon^2 E[v] \end{aligned}$$

where we have used Green's first identity.[1] The minimum occurs at $\varepsilon = 0$ so by calculus, $\int_\Omega v\nabla^2 u \, d\mathbf{x} = 0$. This is valid for almost all functions v with the properties defined above.

Now let D_1, D_2 be two arbitrary nonempty subsets of Ω such that the closure of D_1 is contained completely within the closure of D_2, and the closure of D_2 is contained completely within Ω. Define a special v such that $v \equiv 1$ in D_1, and $v \equiv 0$ in $\Omega \setminus \bar{D}_2$, and $v(\mathbf{x})$ has a smooth transition from 1 to 0 in $D_2 \setminus D_1$. Then $\int_{D_1} \nabla^2 u \, d\mathbf{x} = 0$ for all such D_1. By the second vanishing

[1] Recall that for $\Omega \subset \mathbb{R}^2$ being simply-connected, and $\partial\Omega$ having unit outward normal vector ν, then because of the vector equality $\nabla \cdot (v\nabla u) = \nabla v \cdot \nabla u + v\nabla^2 u$, we have $\int_\Omega v\frac{\partial u}{\partial \nu} \, ds = \int_\Omega \nabla v \cdot \nabla u \, d\mathbf{x} + \int_\Omega v\nabla^2 u \, d\mathbf{x}$.

theorem[2], it follows that $\nabla^2 u = 0$ in Ω, which implies u is harmonic in Ω. By uniqueness of the Dirichlet problem, it is the only harmonic function satisfying (1) that can minimize energy.

[2]Let $f \in C(\Omega)$ and $\int_D f(\mathbf{x})\, d\mathbf{x} = 0$ for all subdomains $D \subset \Omega$. Then $f(\mathbf{x}) \equiv 0$ in Ω.